A Strong Hot Spot Theorem

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In [3] Bailey and Richard Crandall established normality base b for the class of constants

$$\alpha_{b,c} = \sum_{k=1}^{\infty} \frac{1}{c^k b^{c^k}}, \tag{1}$$

where the integer $b > 1$ and c is odd and co-prime to b, as well as some generalizations of this class. The proof given in [3] is rather difficult and relies on several not-well-known results, including one by Korobov on the properties of certain pseudo-random sequences. Recently it has been shown that normality can be established much more easily, as a consequence of what may be called the "hot spot" theorem [1]. Here we state and prove a strong form of the "hot spot" theorem. A weaker result is given in [5, pg. 77], and is proven by an ergodic theory argument in [2].

In the following, $\{\cdot\}$ denotes fractional part as before, and $\#[\cdot]$ denotes count. μ and ν denote probability measures on U (the unit interval mod 1). $A - B$ denotes the set of $x \in A$ and $x \notin B$, and $A \Delta B = (A - B) \cup (B - A)$. The notation a.e. $x[\mu]$ means for all $x \in U$ except for a set Q with $\mu(Q) = 0$. A *Vitali covering* of a measurable set $A \subset U$ is a collection of open intervals with the property that every $x \in A$ is contained in infinitely many, arbitrarily small intervals in the collection. The measure ν is *absolutely continuous* with respect to μ if $\nu(A) = 0$ whenever $\mu(A) = 0$. The map $T : U \to U$ is said to be *measure-preserving* with respect to μ if $\mu(T^{-1}A) = \mu(A)$ for every μ-measurable set A, and *ergodic* with respect to μ if $T^{-1}A = A$ implies $\mu(A) = 0$ or 1.

Given a real constant α in $[0, 1)$, we define here a *base-b hot spot* to be some $x \in [0, 1)$ with the property that

$$\liminf_{h \to 0} \liminf_{n \to \infty} \frac{\#_{0 \le j < n}[\{b^j \alpha\} \in (x - h, x + h)]}{2hn} = \infty. \tag{2}$$

Another way to state this condition is this: x is a base-b hot spot if given any $M > 0$, there is some $\delta_M > 0$ such that for all $h < \delta_M$ there is some $N_h > 0$ such that for all $n > N_h$, the condition $\#_{0 \le j < n}[\{b^j \alpha\} \in (x - h, x + h)] > 2hnM$ holds.

What we shall establish below is that α is b-normal if and only if it has no base-b hot spots. We first present a few preliminary results.

Lemma 1 Vitali covering lemma. *If a μ-measurable set $A \subset U$ has a Vitali covering, then given any $\epsilon > 0$, there is some finite disjoint subcollection A' with the property that $\mu(A \Delta A') < \epsilon$.*

This result is proven in [6].

Lemma 2 Birkoff ergodic theorem. *Let $f(t)$ be an integrable function on $[0,1)$, and let T be an ergodic transformation for μ. Then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k x) \;=\; \int f \, d\mu \quad \textit{for a.e. } x[\mu], \tag{3}$$

This result is proved in [4, pg. 13, 20-29].

Lemma 3 Equivalence of absolutely continuous measures. *Suppose that T is measure-preserving and ergodic with respect to both μ and ν, and further that ν is absolutely continuous with respect to μ. Then $\mu = \nu$.*

Proof. Applying the ergodic theorem to $f(t) = I_A(t)$ (the indicator function of A),

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k x) \;=\; \int f(t)\, d\mu(t) \;= \mu(A) \quad \text{for a.e. } x[\mu]. \tag{4}$$

Since ν is absolutely continuous with respect to μ, the above holds a.e. $x[\nu]$ as well. Now since T preserves the measure ν, we can write, for $n > 0$,

$$\begin{aligned} \nu(A) \;&=\; \int f(t)\, d\nu(t) \;=\; \frac{1}{n} \sum_{i=0}^{n-1} \int f(T^i x)\, d\nu(x) \\ &=\; \int \frac{1}{n} \sum_{i=0}^{n-1} f(T^i x)\, d\nu(x) \;\to\; \int \mu(A) d\nu \;=\; \mu(A), \end{aligned} \tag{5}$$

by the dominated convergence theorem. QED

Lemma 4 Absolute continuity of measures with finite derivatives. *Suppose ν is a measure on U with the property that for a.e. $x[\nu]$,*

$$\liminf_{h\to 0} \frac{\nu(x-h, x+h)}{2h} \;<\; \infty \tag{6}$$

Then ν is absolutely continuous with Lebesgue measure.

Proof. Here μ denotes Lebesgue measure on U, and ν denotes any measure as defined in the hypothesis. Let A be any set with $\mu(A) = 0$, and let $\epsilon > 0$ be given. Then there exists a set Q with $\nu(Q) < \epsilon$ and $M \geq 1$ such that the LHS of (6), as a function of x, is bounded by M except on Q. Further, there exists some open set $A' \supset A$ with $\mu(A') < \epsilon/M$. Then for every $x \in (A' - Q)$ there exists an infinite sequence h_k, strictly decreasing to zero, such that $(x - h_1, x + h_1) \subset A'$ and $\nu(x - h_k, x + h_k)/(2h_k) \leq M + \epsilon$ for $k \geq 1$. For $x \in (A' \cap Q)$, define $h_k = 2^{-m-k}$, where m is large enough that $(x - h_1, x + h_1) \subset A'$. Note that in either case all of these intervals are contained within A'. The collection of these intervals is a Vitali covering of the set A', so by the Vitali covering lemma there is a finite disjoint subcollection $A'' \subset A'$ with $\nu(A' - A'') < \epsilon$. We can then write

$$\begin{aligned}
\nu(A) &\leq \nu(A') \;=\; \nu(A'') + \nu(A' - A'') \\
&= \nu(A'' - Q) + \nu(A'' \cap Q) + \nu(A' - A'') \\
&\leq (M + \epsilon)\mu(A'' - Q) + 2\epsilon \;\leq\; (M + \epsilon)\mu(A') + 2\epsilon \\
&\leq (M + \epsilon)\epsilon/M + 2\epsilon \;<\; 4\epsilon, \qquad (7)
\end{aligned}$$

which implies that $\nu(A) = 0$. QED

In the following, μ will denote Lebesgue measure on U, and, given a real constant $\alpha \in U$ and an integer $b \geq 2$, ν will denote the measure defined on an interval (c, d) as

$$\nu(c, d) \;=\; \liminf_{n \to \infty} \frac{\#_{0 \leq j < n}[\{b^j \alpha\} \in (c, d)]}{n} \qquad (8)$$

Lemma 5 Ergodicity of the digit-shift transformation. *The digit-shift transformation $T(x) = \{bx\}$ is measure-preserving and ergodic with respect to both μ and ν.*

Proof. T clearly preserves Lebesgue measure. Assume for convenience that $b = 2$, and suppose that $A = T^{-1}(A)$. Then note that $x \in A$ if and only if $\{x + 1/2\} \in A$. Thus if $D = (0, 1/2)$, then $\mu(A \cap D) = \mu(A)/2 = \mu(A)\mu(D)$. A similar equality follows for any binary rational interval $(j2^m, k2^m)$, and thus for any finite disjoint union of such intervals. This collection of binary rational intervals is a Vitali covering of A. Thus given $\epsilon > 0$, there is some finite disjoint union E with $\mu(A \Delta E) < \epsilon$ and $\mu(A \cap E) = \mu(A)\mu(E)$. We can then write

$$\begin{aligned}
|\mu(A) - \mu^2(A)| &< |\mu(A) - \mu(A)\mu(E)| + \epsilon \;=\; |\mu(A) - \mu(A \cap E)| + \epsilon \\
&= |\mu(A) - (\mu(A) - \mu(A - E))| + \epsilon \;\leq\; 2\epsilon \qquad (9)
\end{aligned}$$

Thus $\mu(A) = \mu^2(A)$, so that $\mu(A) = 0$ or 1 as required. A similar argument applies to the measure ν as defined above. In the parlance of ergodic theory, T is "mixing" with respect to both μ and ν, which condition is well-known to imply ergodicity [4, pg. 12]. QED

Theorem 1 Hot spot theorem. *The real constant α is b-normal if and only if it has no base-b hot spots.*

Proof. If α has no base-b hot spots, then it follows immediately from Lemmas 3, 4, and 5, that for any interval $(c, d) \subset U$,

$$\liminf_{n \to \infty} \frac{\#_{0 \leq j < n}[\{b^j \alpha\} \in (c, d)]}{n} \;=\; \mu(c, d) \;=\; d - c \qquad (10)$$

This result also applies to $(0, c) \cup (d, 1)$, which except for c, d and the point 0 is the complement of (c, d). We can then write

$$\begin{aligned}\limsup_{n\to\infty} \frac{\#_{0\le j<n}[\{b^j\alpha\} \in (c,d)]}{n} &= 1 - \liminf_{n\to\infty} \frac{\#_{0\le j<n}[\{b^j\alpha\} \in (0,c) \cup (d,1)]}{n} \\ &= 1 - (c + (1-d)) \;=\; d - c \end{aligned} \tag{11}$$

Thus the lim inf and the lim sup are identical. Since this holds for any interval (c, d), it holds in particular for any interval whose endpoints are of the form j/b^m. Thus α is b-normal. QED

References

[1] David H. Bailey, "A Hot-Spot Proof of Normality for the Alpha Constants," manuscript, 2003, available at the URL
`http://www.nersc.gov/~dhbailey/dhbpapers/alpha-normal.pdf`

[2] David H. Bailey and Daniel J. Rudolph, "An Ergodic Proof that Rational Times Normal is Normal," manuscript, 2002, available at the URL
`http://www.nersc.gov/~dhbailey/dhbpapers/ratxnormal.pdf`

[3] David H. Bailey and Richard E. Crandall, "Random Generators and Normal Numbers," *Experimental Mathematics*, vol. 11 (2002), pg. 527–546.

[4] Patrick Billingsley, *Ergodic Theory and Information*, John Wiley, New York, 1965.

[5] L. Kuipers and H. Niederreiter, *Uniform Distribution of Sequences*, Wiley-Interscience, New York, 1974.

[6] Halsey L. Royden, *Measure Theory*, Addison-Wesley, 1968.